Multiloop Feynman Integrals
in the Worldline Approach *

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Abstract

We explain the concept of worldline Green functions on classes of multiloop graphs. The QED β – function and the 2-loop Euler-Heisenberg Lagrangian are discussed for illustration.

Introduction

The first quantized approach to quantum field theory (QFTH) involving the worldline path integral of relativistic particles is well known in the case of propagators and one-loop effective actions. Still its consequent use as an alternative to Feynman diagram calculations has been pursued only recently [1] (compare our contribution [2] and references quoted therein): This was triggered by a systematic discussion of the string tension $\frac{1}{\alpha'} \to \infty$ limit of string theory [3]. The latter is also usually formulated as a first quantized theory and it should lead to local QFTH in this limit. Indeed the relativistic particle quantization on the worldline was used to argue for a σ model formulation of string theory on the worldsheet.

Surprisingly the extension of the worldline formulation to the multiloop case is much less developed. Starting from the multiloop string amplitudes [4] and discussing $\frac{1}{\alpha'} \to \infty$ looks like being the most straightforward procedure but unfortunately this is technically very ambitious. One can also use a separate worldline path integral for each internal propagator [5] or a stringlike reorganization of standard Feynman parameter integrals [6].

We proposed [7] a multiloop generalization of Strassler's [1] approach based on the concept of worldline Green functions. It is known that in the case of multiloop string amplitudes there exists a much smaller number of topologically different diagrams per loop order than in ordinary field theory since the vertices can be deformed continuously. This is very appealing since it has the prospect of writing down master expressions for whole classes of Feynman graphs of a given loop order. The existence of a worldline Green function characteristic for such a class of Feynman graphs is essential. Because of the inner vertices these Green functions are not defined on a one-dimensional manifold (like for the circle) and hence their existence is nontrivial.

2-loop Green functions

We start from the simple 1-loop effective action in ϕ^3 theory (mass m) which can be written as a path integral over the circle (see eqs.(1),(2) of [2]). Correlating two of the background field insertions at $x(\tau_a), x(\tau_b)$ by a Feynman propagator we obtain a 2-loop expression. The propagator which we take to be the free one for the beginning can be written as a Feynman-Schwinger path integral over $\bar{x}(\bar{\tau}) = (x_a(\bar{T} - \bar{\tau}) + x_b\bar{\tau})/\bar{T} + \bar{y}(\bar{\tau})$ with $\bar{x}(0) = x_a = x(\tau_a)$ and $\bar{x}(\bar{T}) = x_b = x(\tau_b)$ or directly as the well-known propagator in x – space (euclidean dimension D)

$$G_F(x_b - x_a) = \int_0^\infty \frac{d\bar{T}}{(4\pi\bar{T})^{D/2}} \exp\left\{ \frac{(x_b - x_a)^2}{4\bar{T}} - m^2\bar{T} \right\}. \tag{1}$$

Then the integral

$$\int_0^\infty \frac{dT}{T} \int_0^\infty d\bar{T} \frac{1}{(4\pi\bar{T})^{D/2}} e^{-m^2(T+\bar{T})} \int_0^T d\tau_a d\tau_b \int [Dx(\tau)]$$

$$\times \exp\left[-\int_0^T d\tau \left(\frac{\dot{x}_\mu^2}{4} + \frac{(x(\tau_a) - x(\tau_b))_\mu^2}{4T\bar{T}} + \lambda\phi(x(\tau))\right)\right] \qquad (2)$$

corresponds to a certain linear combination of all 2-loop Feynman diagrams with the external legs sitting on the loop but not on the inner propagator.

Since $x(\tau_a), x(\tau_b)$ appear quadratically in the exponential, they can be included in the Gaussian part of the path integral. We can obtain a new Green function with the inner line taken into account by inverting the operator P,

$$P_{\tau'\tau} = \delta(\tau' - \tau)\frac{\partial_\tau^2}{2} - \frac{1}{2\bar{T}}(\delta(\tau' - \tau_a) - \delta(\tau' - \tau_b))(\delta(\tau_a - \tau) - \delta(\tau_b - \tau)) \qquad (3)$$

as

$$G_B^{(1)}(\tau, \tau') = G_B(\tau, \tau') + \frac{1}{2}\frac{[G_B(\tau, \tau_a) - G_B(\tau, \tau_b)][G_B(\tau_a, \tau') - G_B(\tau_b, \tau')]}{\bar{T} + G_B(\tau_a, \tau_b)} \qquad (4)$$

where $G_B(\tau, \tau') = |\tau - \tau'| - \frac{(\tau-\tau')^2}{T}$ is the Green function on the circle discussed in [1],[2]. We also obtain a new Gaussian path integration determinant,

$$(4\pi T)^{-1/2}[1 + \frac{1}{\bar{T}}G_B(\tau_a, \tau_b)]^{-D/2} \qquad (5)$$

(with the first factor being the 1-loop determinant). Contributions to the n-point scattering amplitude are obtained from (2) by expanding for a plane wave background $\phi = \sum_{j=1}^n e^{ip_j x(\tau)}$. Contraction of the $e^{ip_j x(\tau_j)}$ with the new Green function (4) leads to the following integral [7]

$$\int_0^\infty \frac{dT}{T} \int_0^\infty d\bar{T} e^{-m^2(T+\bar{T})}(4\pi)^{-D} \int_0^T d\tau_a d\tau_b [T\bar{T} + TG_B(\tau_a, \tau_b)]^{-D/2}$$

$$\times \prod_{i=1}^n \int_0^T d\tau_i \exp[\sum_{k<l} G_B^{(1)}(\tau_k, \tau_l)p_k p_l + \frac{1}{2}\sum_{k=1}^n G_B^{(1)}(\tau_k, \tau_k)p_k^2] \qquad (6)$$

We have compared [7],[9] this with the ordinary Feynman diagram parametrization: The α parameters correspond to differences of τ parameters, and each Feynman graph can be recast into the form (6) with the τ-integration restricted to a certain ordered sector. Thus the new finding is that the Feynman parameter expressions looking completely different for different Feynman graphs can be written as parts of one master expression.

If there are also background couplings to the "inner line", this can be taken into account by using a propagator in the background. We then also have contractions of the inner variable $\bar{y}(\bar{\tau})$ mentioned above. In the particular case of ϕ^3 a symmetric description in all three propagator lines is preferable and such a Green function, now valid for contractions all over the 2-loop graph, was given by the present authors [7].

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We should emphasize that the worldline Green functions are not characteristic for a particular QFTH but just for the topology of the class of graphs. In the case of more complicated QFTH's like QED worldline vertex factors have to be included (and contracted). This was the main issue in [2].

In the case of multiloop diagrams constructed from a closed loop by connecting $2m$ points by propagators, worldline Green functions can be written down along the same lines [7]. We found

$$G_B^{(m)} = G_B(\tau, \tau') + \frac{1}{2} \sum_{k,l=1}^{m} [G_B(\tau, \tau_{a_k}) - G_B(\tau, \tau_{b_k})] A_{kl}^{(m)} [G_B(\tau_{a_l}, \tau') - G_B(\tau_{b_l}, \tau')] \quad (7)$$

where the $m \times m$ matrix $A^{(m)}$ is defined by

$$(A^{(m)})^{-1} = \bar{T} - \frac{1}{2} B; \ \bar{T}_{kl} = \bar{T}_k \delta_{kl}; \ B_{kl} = G_{B_{a_k,a_l}} - G_{B_{a_k,b_l}} - G_{B_{b_k,a_l}} + G_{B_{b_k,b_l}}. \quad (8)$$

The determinant factor is $\text{Det}(A^{(m)})^{D/2}$. Also the case with two closed electron loops connected by photons can be handled easily [10] (see also [11]).

It is nice to see that our expressions (7), (8) could be confirmed and generalized recently by Roland and Sato [12] by an explicit discussion of the limit $\frac{1}{\alpha'} \to \infty$ in string theory, i.e. considering a two-punctured genus $(m+1)$ Riemann surface. It turns out that the multiloop worldline Green functions as given above are just the leading order terms of the string theory Green functions in this limit.

Applications: 2- and 3-loop QED β-function, 2-loop Euler-Heisenberg Lagrangian

The scalar and spinor QED β-functions in 2 and higher loop order are interesting objects: Complicated calculations have simple results (no $\zeta(2n+1)$ terms, cancellation of subdivergences). The 2-loop exercise can be found in [8] in one version, but there are more efficient ones [10]. For this purpose, we introduce a constant external field $\mathbf{F}$. In the Fock-Schwinger gauge (see [2]) $A_\mu = \frac{1}{2} y^\rho F_{\rho\mu}$ such a field may be absorbed into the worldline Green function on the circle [13],[14]

$$\mathcal{G}_B(\tau_1, \tau_2) = \frac{1}{2(e\mathbf{F})^2} \left(\frac{e\mathbf{F}}{\sin(e\mathbf{F}T)} e^{-ie\mathbf{F}\dot{G}_{B_{12}}T} + ie\mathbf{F}\dot{G}_{B_{12}} - \frac{1}{T} \right) \quad (9)$$

The path integral determinant becomes $(4\pi T)^{-D/2} \exp\{-\frac{1}{2}\text{tr}\ln\frac{\sin e\mathbf{F}T}{e\mathbf{F}T}\}$. This allows for a very efficient evaluation of scattering amplitudes in a constant background field, e.g. photon splitting in a constant magnetic field [15].

Considering the scalar QED β-function calculation, to obtain the β – function coefficient we need to expand the effective action only to second order in $\mathbf{F}$. To take an internal photon into account, the denominator of its x – space propagator (in Feynman gauge) is treated like the scalar propagator, and we have almost the ϕ^3 case except that one has to contract also vertex factors $\dot{x}_\mu(\tau_{a,b})$. The 2-loop Green function (4) is now written with the new $\mathcal{G}_B(\tau, \tau')$ instead of the original G_B, and a denominator $\bar{T} - \frac{1}{2}(\mathcal{G}_{aa} + \mathcal{G}_{bb} - \mathcal{G}_{ab} - \mathcal{G}_{ba})$ instead of $\bar{T} + G(\tau_a, \tau_b)$ since there are "self

contractions". The latter is also true for the determinant factor (5). The resulting expression can be easily integrated and the singular part can be extracted. Note that the use of dimensional regularization is very natural in this approach.

Counterterms still are formed like in conventional QFTH. We hope to finally embed this in a worldline procedure. Note that the gauge freedom for the photon propagator is hidden in a total τ – derivative.

To generalize this from the scalar to the spinor loop, we have just to substitute superquantities everywhere, and to write down also the fermionic analogue of (9). It is satisfying that even the multiloop Green functions (7) and the determinant can be generalized to superquantities [8].

For the 3-loop QED β – function (one electron loop: "quenched" part) we [16] can use formulas (7), (8) for $m = 2$. Just for an example, the most basic parameter integral which we have to solve is

$$I = \int_0^\infty dT_1 dT_2 \int_0^1 d\tau_a d\tau_b d\tau_c d\tau_d [(T_1 + G_{B_{ab}})(T_2 + G_{B_{cd}}) - C^2/4]^{-D/2} \qquad (10)$$

with $C = G_{B_{ac}} - G_{B_{ad}} - G_{B_{bc}} + G_{B_{bd}}$ and $\tau_{a,b}$ and $\tau_{c,d}$ the end positions of the two inner photons on the electron loop.

The electron proper time integral $\int_0^\infty \frac{dT}{T} e^{-m^2 T} T^{6 - \frac{3}{2}D} = \Gamma(6 - \frac{3}{2}D) m^{3D-12}$ decouples and just gives the overall $\frac{1}{\epsilon}$ pole. The singular part of (10) is easy to split off and to calculate, the regular part can be evaluated at $D = 4$. The variable C can be integrated out trivially leading to known 2-parameter integrals.

The old Euler-Heisenberg-Schwinger formula for the 1-loop effective action in a constant electromagnetic field F is immediately obtained from the above 1-loop path integral determinant. This can be extended to the calculation of the 2-loop correction to this effective action due to one photon exchange, as should be already clear from what we said about the 2-loop QED β-function calculation. In this case we keep all orders in F and we are interested in the finite part. Renormalization can be performed and we obtain [13] the known result in an elegant way.

References

[1] M. J. Strassler, Nucl. Phys. **B385** (1992) 145

[2] M. G. Schmidt, C. Schubert, "Relativistic QFTH Couplings on the Worldline", contribution to this conference

[3] Z. Bern, D. A. Kosower, Nucl. Phys. **B379** (1992) 451

[4] K. Roland, Phys. Lett. **B289** (1992) 148

[5] D. G. C. McKeon, Ann. Phys. (N.Y.) **224** (1993) 139

[6] C. S. Lam, Phys. Rev. **D48** (1993) 873

[7] M. G. Schmidt, C. Schubert, Phys. Lett. **B331** (1994) 69

[8] M. G. Schmidt, C. Schubert, Phys. Rev. **D53** (1996) 2150

[9] See also H.-T. Sato, Phys. Lett. **B371** (1996) 270

[10] M. G. Schmidt, C. Schubert, forthcoming review

[11] K. Daikouji, M. Shino, Y. Sumino, Phys. Rev. **D53** (1996) 4598

[12] K. Roland, H.-T. Sato, NBI-HE-96-19 (hep-th/9604152)

[13] M. Reuter, M. G. Schmidt, C. Schubert, IASSNS-HEP-96/90 (hep-th/9610191)

[14] D. Cangemi, E. D'Hoker, G. Dunne, Phys. Rev. **D51** (1995) 2513;
R. Zh. Shaisultanov, Phys. Lett. **B378** (1996) 354 (hep-th/9512142)

[15] S. L. Adler, C. Schubert, Phys. Rev. Lett. **77** (1996) 1695 (hep-th/9605035)

[16] D. Fliegner, M. G. Schmidt, C. Schubert, work in progress;
HD-THEP-96-18 (hep-th/9608175) (Rheinsberg conference 1996 proceedings)